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Reviewed work(s):
Source: *Econometrica*, Vol. 40, No. 1 (Jan., 1972), pp. 147-157
Published by: The Econometric Society
Stable URL: http://www.jstor.org/stable/1909727

Econometrica, Vol. 40, No. 1 (January, 1972)

SOCIAL PREFERENCE ORDERINGS AND MAJORITY RULE[1]

BY OTTO A. DAVIS, MORRIS H. DEGROOT, AND MELVIN J. HINICH

1. INTRODUCTION

IMAGINE A SOCIETY consisting of a finite number of individuals, each of whom has a transitive preference ordering defined over the elements in a set of alternatives. In a classic work Arrow [**1**] proved that if the number of alternatives is greater than two, it is impossible in general to create a group or social preference ordering which satisfies five reasonable conditions. However, if the set of alternatives is one-dimensional, if the individual preference orderings are single peaked, and if the number of individuals is odd, then Black [**3**] proved that in any finite set of alternatives there will be one which will command a majority over any other. Indeed, under these assumptions Black demonstrated that the principle of majority rule would lead to a transitive ordering of the alternatives. Arrow extended this result to show that, no matter how many alternatives there are in the set, when the total number of individuals is odd and they all have single-peaked preferences then simple majority rule does yield a social preference ordering which is both transitive and satisfies Arrow's five conditions.

Black introduced and considered the conception of single-peakedness within the context of a single dimension. While Arrow [**1**, pp. 75–6] argued that the concept of single-peakedness was inherently one-dimensional, he proposed a formal definition of the concept which basically involves a condition over triples, and Inada [**6**] demonstrated that utility functions which satisfy the formal definition need not conform to the more intuitive notion of unimodality, even though the latter probably does illustrate the motivation of both Black and Arrow. The utility functions considered in this paper are unimodal and multidimensional, but the class of restrictions which they satisfy does not translate itself into conditions on triples so that, strictly speaking, they need not satisfy the formal definition of single-peakedness.

Since the publication of the books of both Arrow and Black, there has been much work on the problem of social choice. Inada [**6**], Sen [**10**], and Pattanaik [**7**] are concerned with classes of restrictions that are wider than single-peakedness. Sen [**9**] has also examined the relationship between preferences and voting when the utility or disutility of voting is introduced into the analysis in order to handle abstentions.

This paper takes a fresh look at the problem of social choice. The problem is formulated in a manner that differs importantly from that utilized by Arrow and the references cited above. This new formulation, which is based upon that developed in Davis and Hinich [**4**] and the earlier references indicated there, provides additional insight. It allows certain geometric interpretations and also provides

[1] This work was supported by a grant from Resources for the Future and by the National Science Foundation under grant GP 8824. The authors are indebted to Professor Charles R. Plott, California Institute of Technology, and an anonymous referee for some very useful comments.

recourse to the tools of the probability calculus. In so doing it suggests that it may be necessary to include in our analysis certain considerations which have been largely absent from the discussions in the literature if our understanding of the problem of social choice is to advance beyond its present state.

In certain respects, the works of Plott [**8**], Tullock [**12**], and Simpson [**11**] are more closely related to the research reported here than the other literature cited. Similar to each of these papers, this approach represents the alternatives available to society by defining them in terms of points in n-dimensional space. Given this conception and a well defined preference function for each individual, a special kind of majority rule (probably best called non-minority rule) is specified, and Arrow's famous paradox is illustrated in terms of a simple example. Necessary and sufficient conditions are defined for the existence of a unique alternative that will receive a majority against any other alternative. These conditions for the dominance of a single alternative are closely related to the results of Plott [**8**] who, using a different formulation which allows only a finite number of individuals to be considered, also explores the problem of determining when a unique alternative can be certain to command a majority. Tullock [**12**] also is concerned solely with dominance, but his two-dimensional structure bears a close resemblance to the n-dimensional one used here. In particular, the utility functions implicit in Tullock's analysis, which is informally developed without theorems or proofs by the device of insightful examples, belong to the same general class as the ones utilized herein. Simpson [**11**] combines the approaches of Plott and Tullock to obtain conditions for the dominance of a unique alternative when the number of individuals is finite.

In addition to the issue of dominance, there is also the problem of finding conditions under which an unambiguous social preference ordering can be defined. While the above do not consider this problem, necessary and sufficient conditions are developed herein for majority rule to define a transitive social preference ordering for the stated class of utility functions.

2. THE STRUCTURE

Suppose there exists a set of n cardinal dimensions of choice such that every alternative can be uniquely mapped into the Euclidean n-space which has these dimensions as the axes of the coordinate system. We assume that all points $x \in E_n$ (viewed as column vectors $x' = (x_1, \ldots, x_n)$) are possible alternatives and that all individuals perceive the alternatives in a common manner. For example, educational policy can be measured by the expenditure per pupil, the teacher-pupil ratio, the amount of audio-visual aids used per pupil, etc. We must assume that each individual in the society perceives the same set of dimensions of choice and locates the alternatives in the Euclidean space similarly. These assumptions do *not* imply that all individuals have the same preference orderings defined over the space.

The preference ordering over E_n for any individual is determined by the utility function of the individual. It is convenient to assume that each individual has a most preferred position $x \in E_n$ so that his utility function is maximized by x. While for basically economic dimensions this assumption might appear to be contrary to the usual presumption that individuals have insatiable wants, it is analytically convenient. Further, if one desires to use the concept of loss functions, then one can follow the procedures outlined in Barr and Davis [**2**] so that by beginning with traditional utility functions and budget constraints a reflection of the utility functions defined below is easily obtained. We also assume that there exists a positive definite matrix A such that given an individual i whose preferred position is x, his utility for any other position $y \in E_n$ is $u_i(\|y - x\|_A)$, where u_i is a strictly decreasing function of its argument and

$$\|y - x\|_A^2 = (y - x)'A(y - x).$$

With no loss of generality one can let $A = I$, the $n \times n$ identity matrix, since there exists a linear transformation of the axes which makes utility constant on spheres. With this choice of A, we have

$$\|y - x\|_A^2 = \|y - x\|^2 = (y - x)'(y - x).$$

If the preferred position of an individual is x, then one alternative y will be preferred to another alternative z if and only if

$$u_i(\|y - x\|) > u_i(\|z - x\|)$$

or equivalently, if and only if

$$\|y - x\| < \|z - x\|, \tag{1}$$

since u_i is assumed to be strictly decreasing. If

$$\|y - x\| = \|z - x\|,$$

the individual is indifferent between y and z. For the ith individual, the function u_i transforms Euclidean distance between points in E_n into utiles for the different alternatives.

Although the individuals have the same set of orthogonal choice dimensions, they will not in general have the same most preferred point. As a model of the differences in tastes within the population, let P^* denote the distribution of most preferred points of the individuals. Let X be the most preferred point of an individual chosen at random from the population. Given a (Borel) set $S \subset E_n$, $\Pr(S)$ will denote the probability that $X \in S$ under the distribution P^*.

DEFINITION 1: For any points $y \in E_n$ and $z \in E_n$, it is said that yRz if $\Pr(\|y - X\| \leqslant \|z - X\|) \geqslant \frac{1}{2}$. In other words, yRz if and only if at least half the population either prefers y to z or is indifferent between y and z. If yRz but not zRy, we say that y is preferred to z by majority rule and write yPz. It follows that yPz if and only if $\Pr(\|y - X\| < \|z - X\|) > \frac{1}{2}$. If yRz and zRy, then neither position is preferred to the other by majority rule and we write yIz.

The relation R is reflexive since yRy. Moreover, for points $y \in E_n$ and $z \in E_n$, exactly *one* of the following relations holds: yPz, zPy, or yIz. It is important to note, however, that the special place of indifference in the definition of the relation R

means that it does not completely correspond to the traditional definition of majority rule, at least in certain exceptional circumstances, so that it probably could be more appropriately termed nonminority rule.[2] Nevertheless, since the circumstances for non-correspondence with the traditional conception are clearly exceptional, the usual terminology is used here.

If $n = 1$, it can be shown that the relation R is transitive. However, if $n \geqslant 2$ the relation R is not necessarily transitive, although the functions u_i are unimodal and well behaved, so that the resulting social preference ordering is not necessarily transitive. We shall demonstrate this fact by a simple geometric argument.

First let us present a geometric interpretation of the rule of social preference given by the inequality (1). Let the points $y, z \in E_n$ be fixed. For any point $x \in E_n$, some simple algebra shows that

$$(2) \qquad \|y - x\| \gtreqqless \|z - x\| \qquad \text{if and only if} \qquad (z - y)'\left(x - \frac{y + z}{2}\right) \gtreqqless 0.$$

Therefore, yRz if and only if

$$(3) \qquad \Pr\left[(z - y)'\left(X - \frac{y + z}{2}\right) \leqslant 0\right] \geqslant \tfrac{1}{2}.$$

Geometrically, for any non-zero vector $a \in E_n$ and any real number b, the set of points satisfying the equation $a'x - b = 0$ is a *hyperplane*. Let $w \in E_n$ be any point such that $a'w - b \neq 0$ so that w does *not* lie on the hyperplane $a'x - b = 0$. Any point $v \in E_n$ such that

$$(4) \qquad (a'v - b)(a'w - b) > 0$$

is said to be on the same *side* of the hyperplane as w, or more simply, on the w-side of the hyperplane. The set of all points lying either on the w-side of the hyperplane or on the hyperplane itself will be called the *closed* w-side of the hyperplane. Now let $a = z - y$ and $b = (\frac{1}{2})(z - y)'(y + z)$. From (2) and (3) it follows that yRz if and only if the probability of the closed y-side of the hyperplane $\|y - x\| = \|z - x\|$ is at least $\frac{1}{2}$.

Given this geometric interpretation, let us consider a two-dimensional example to demonstrate that, in general, transitivity does not exist for a multidimensional space E_n even when all utility functions are unimodal. Consider Figure 1 where the horizontal and vertical axes measure the two relevant issues. Assume that the population consists of three individuals and that their most preferred positions

[2] We are indebted to an anonymous referee for pointing out this fact and for supplying the following example, which is quoted with only a little paraphrasing. Consider a case where fifty-one people are indifferent between y and z and forty-nine people strictly prefer z to y. Under majority rule z will, of course, defeat y, but in this definition we shall have yIz, because we have yRz since more than half the population is indifferent between y and z. Thus the social choice rule studied is not really the traditional majority rule, but something like non-minority rule, closely related to R_{maj} as defined by Dummet and Farquharson [**5**].

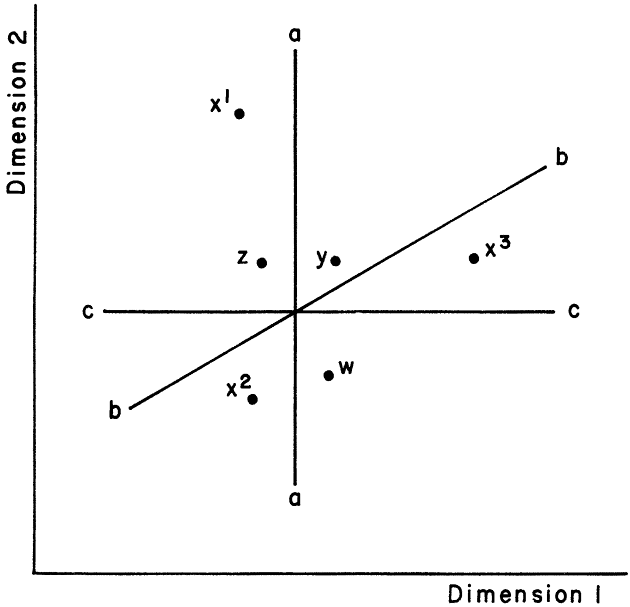

FIGURE 1

are x^1, x^2, and x^3 respectively. For the purpose of this example it is sufficient to limit the analysis to the three alternatives w, y, and z. The hyperplane $\|w - x\| = \|y - x\|$ is the line cc, the hyperplane $\|y - x\| = \|z - x\|$ is the line aa, and the hyperplane $\|z - x\| = \|w - x\|$ is the line bb. From the figure one can see that zPy because two points x^1 and x^2 lie on the z-side of the line aa while only one point x^3 lies on the y-side of the line. Similarly, it is seen from the figure that yPw and wPz. Hence, for this example the relation P is intransitive and thus so is the relation R.

3. DOMINANCE

It is obvious that situations may exist where a single alternative may command at least one half of the votes against all other alternatives even though the social preference ordering is not transitive.

DEFINITION 2: A point y^* is said to be *dominant* if y^*Rz for all $z \in E_n$.

In the next lemma we give a necessary and sufficient condition for a point to be dominant.

LEMMA 1: *A point y^* is dominant if and only if, for every point $a \in E_n$ and for every number $b > 0$,*

(5) $\Pr[a'(X - y^*) \leqslant b] \geqslant \frac{1}{2}$.

PROOF: Suppose first that y^* is dominant, and consider a given point $a \in E_n$ and a given number $b > 0$. We must show that (4) is satisfied for these values. If $a = 0$, then (4) is obviously satisfied. If $a \neq 0$, then let z be the point in E_n defined by (5), $z = y^* + (2b/a'a)a$. Since y^* is dominant, y^*Rx, which means that relation (3) is satisfied. However, when z is specified by equation (5), the relation (3) can be reduced to (4) by simple algebra.

Conversely, suppose that (4) is satisfied for every point $a \in E_n$ and every number $b > 0$. For any point $z \in E_n$, relation (3) is of the same form as relation (4) with $y = y^*$, $a = z - y^*$, and $b = \frac{1}{2}\|y^* - z\|^2$. Therefore, y^*Rz. It follows that y^* is dominant.

The interpretation of Lemma 1 is that a point y^* is dominant if and only if, for any hyperplane on which y^* does not lie, the probability of the closed y^*-side of the hyperplane is at least $\frac{1}{2}$. The next theorem states that y^* is dominant if and only if, for any hyperplane containing y^*, the probability of the set of points lying either on the hyperplane or on one side of it is at least $\frac{1}{2}$, and this property is true for both sides of every such hyperplane.

THEOREM 1: *A point y^* is dominant if and only if, for every point $a \in E_n$,*

$$\Pr[a'(X - y^*) \leqslant 0] \geqslant \tfrac{1}{2}. \tag{6}$$

PROOF: Suppose y^* is dominant. Then (4) is satisfied for every $a \in E_n$ and for every $b > 0$. If we let $b \to 0$ in (4), then (6) is obtained. Conversely, suppose (6) is satisfied for every $a \in E_n$. Then a fortiori (4) is also satisfied for every $b > 0$. Thus, by Lemma 1, y^* is dominant.

It should be noted that instead of (6) in Theorem 1, we could just as well have required that for every point $a \in E_n$,

$$\Pr[a'(X - y^*) \geqslant 0] \geqslant \tfrac{1}{2}. \tag{7}$$

COROLLARY 1: *For $n = 1$, any median of the distribution P^* is dominant and only medians are dominant.*

PROOF: By definition, the number y^* is a median of P^* if and only if both $\Pr(X \leqslant y^*) \geqslant \frac{1}{2}$ and $\Pr(X \geqslant y^*) \geqslant \frac{1}{2}$. Thus, for every real number a, $\Pr[a(X - y^*) \leqslant 0] \geqslant \frac{1}{2}$. The corollary follows from Theorem 1.

While a median (and thus dominance) always exists in a single dimension, in higher dimensions ($n \geqslant 2$) a dominant point may not exist. For example, there is no dominant alternative for the population illustrated by Figure 1. Using a related structure, Plott [**8**] explores the special nature of dominance in social choice. Since the structures are somewhat different, there may be some interest in continuing to explore the concept of dominance in the present context.

DEFINITION 3: For any point $y \in E_n$ and any (Borel) set $A \subset E_n$, let Ay be the reflection of the set A through the point y; i.e., let

(8) $$Ay = \{2y - x : x \in A\}.$$

The distribution P^* is said to be *symmetric about the point y* if for every (Borel) set $A \subset E_n$

(9) $$\Pr(Ay) = \Pr(A).$$

In particular, if the distribution P^* can be represented by either a probability density function f or a discrete frequency function f on E_n, then P^* is symmetric about y if $f(x) = f(2y - x)$ for all $x \in E_n$ with the possible exception of a set of probability zero.

THEOREM 2: *If P^* is symmetric about y^*, then y^* is dominant.*

PROOF: Since P^* is symmetric about y^*, it follows from (8) and (9) that, for every point $a \in E_n$,

$$\Pr[a'(X - y^*) < 0] = \Pr[a'(X - y^*) > 0].$$

Therefore, (6) is satisfied and y^* is dominant.

The following are some examples of distributions which are symmetric about some point y^*.

(i) A discrete distribution on a set of $2k + 1$ points in $E_n\{0, x_1, -x_1, \ldots, x_k, -x_k\}$ such that $f(x_i) = f(-x_i)$ for $i = 1, \ldots, k$. For this distribution, $y^* = 0$.

(ii) A multivariate normal distribution with mean μ and non-singular covariance matrix Σ. For this distribution $y^* = \mu$.

(iii) The probability density f on E_n defined by $f(x) = \frac{1}{2}[f_1(x) + f_2(x)]$, where f_1 is a multivariate normal density with mean μ_1 and non-singular covariance Σ and f_2 is a multivariate normal density with a different mean μ_2 but with the same covariance Σ. For this distribution $y^* = \frac{1}{2}(\mu_1 + \mu_2)$.

It should be pointed out that symmetry of this type is a sufficient condition for the existence of a dominant point, but it is not necessary. We have already seen (Corollary 1) that the median of any unidimensional distribution is dominant. In any number of dimensions, if a single point carries probability $p \geqslant \frac{1}{2}$, it is dominant regardless of how the remaining probability is distributed.

4. TRANSITIVITY AND DOMINANCE

At the beginning of the previous section it was pointed out that the existence of a dominant point was not sufficient for the social preference ordering to be transitive. Nevertheless, there is a relationship between dominance and the transitivity of the relation R, and this section is devoted to the exploration of this relationship.

THEOREM 3: *Suppose there exists a dominant point y^*. For any two points $y \in E_n$ and $z \in E_n$, if $\|y - y^*\| \leqslant \|z - y^*\|$, then yRz.*

PROOF: Consider the hyperplane of points x for which $\|y - x\| = \|z - x\|$. If $\|y - y^*\| \leqslant \|z - y^*\|$, then y^* must lie on the closed y-side of this hyperplane. Thus it follows from Lemma 1 and Theorem 1 that the probability of the closed y-side of the hyperplane is at least one half. Hence, yRz.

One is tempted to think, at least at first glance, that the assumption that there exists a dominant point and the consequences of Theorem 3 are sufficient to guarantee that the relation R is transitive. Unfortunately, these are not quite sufficient, as the following example demonstrates. Imagine a two-dimensional space and consider the three alternatives w, y, and z which are shown in Figure 2. Suppose that the probability distributed P^* assigns probability $\frac{1}{2}$ to the point w, assigns probability $\frac{1}{2}$ to the small region A distributed with a uniform density over the region, and assigns probability zero to the rest of the space. Then w is dominant in this example. Furthermore, yRz and zRw (in fact, yIz and zIw), but wPy. Hence, the relation is not transitive.

In order to obtain the transitivity of R, we now introduce a further condition on P^*.

DEFINITION 4: The distribution P^* is said to have a *unique median in all directions* if, for every point $a \in E_n$ ($a \neq 0$), there is a *unique* number b such that both

$$\Pr(a'X \leqslant b) \geqslant \tfrac{1}{2} \qquad \text{and} \qquad \Pr(a'X \geqslant b) \geqslant \tfrac{1}{2}.$$

It is rather obvious, of course, that for any given choice of a, there always exists at least one value of b which satisfies the conditions. What is involved here, however, is the assumption that each of the random variables $a'X$ has a *unique* median. For example, the mean μ of a multivariate normal distribution is a unique

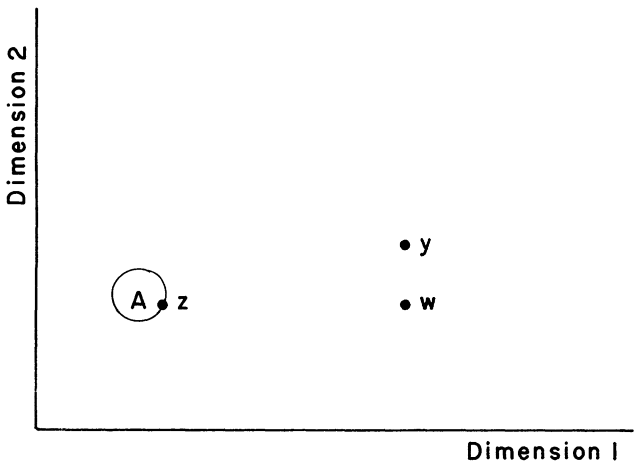

FIGURE 2

median in all directions since $a'X$ has a unique median $a'\mu$ for every a. More generally, if the distribution P^* has a positive density function over the entire space E_n, then the random variable $a'X$ will have a unique median for every value of a.

THEOREM 4: *Suppose that the distribution P^* has a unique median in all directions, and suppose that there exists a dominant point y^*. Then for any two points $w \in E_n$ and $v \in E_n$, $\|w - y^*\| < \|v - y^*\|$ if and only if wPv.*

PROOF: Suppose $\|w - y^*\| < \|v - y^*\|$. Then y^* lies on the w-side of the hyperplane $\|w - x\| = \|v - x\|$. Since y^* is dominant, the probability of the set of points on the closed w-side of this hyperplane is at least $\frac{1}{2}$. We shall now show that the probability of the set of points on the closed v-side of this hyperplane is strictly less than $\frac{1}{2}$, so wPv.

Let the hyperplane being considered be represented by the equation $a'X = b$. Consider a parallel hyperplane which passes through y^*. Obviously, this hyperplane can be represented by the equation $a'X = a'y^*$. Since y^* is dominant, it follows from (6) and (7) that

$$\Pr(a'X \leqslant a'y^*) \geqslant \tfrac{1}{2} \quad \text{and} \quad \Pr(a'X \geqslant a'y^*) \geqslant \tfrac{1}{2}.$$

Since $a'y^* \neq b$ and P^* has a unique median in all directions, it follows that it cannot also be true that both

$$\Pr(a'X \leqslant b) \geqslant \tfrac{1}{2} \quad \text{and} \quad \Pr(a'X \geqslant b) \geqslant \tfrac{1}{2}.$$

One of these sets contains the points on the closed w-side of the hyperplane, and we know that this set has probability $p \geqslant \frac{1}{2}$. The other set, which therefore must have probability less than $\frac{1}{2}$, contains the points on the closed v-side of the hyperplane.

To prove the other half of the theorem suppose now that wPv. Thus, it is not true that vRw and, therefore, by Theorem 3, $\|w - y^*\| < \|v - y^*\|$.

This result allows us to get at the crux of the issue of establishing a transitive social preference ordering from given individual orderings.

COROLLARY 2: *Suppose the distribution P^* has a unique median in all directions, and there exists a dominant point y^*. Then the relation R is transitive and completely orders all of the points in E_n.*

PROOF: For any points $w \in E_n$ and $v \in E_n$, it follows from Theorem 4 that wRv if and only if $\|w - y^*\| \leqslant \|v - y^*\|$. Since Euclidean distance from y^* defines a transitive complete ordering of the points in E_n the corollary follows.

COROLLARY 3: *Suppose the distribution P^* has a unique median in all directions, and there exists a dominant point y^*. Then this dominant point is unique.*

PROOF: Suppose there were another dominant point $y_1 \neq y^*$. Since $\|y^* - y^*\| = 0 < \|y_1 - y^*\|$, it would follow that $y^* P y_1$, and this would violate the assumption that y_1 is dominant.

The following corollary combines Theorem 2 and Corollary 2.

COROLLARY 4: *Suppose that the distribution P^* is symmetric about a point $y^* \in E_n$ and suppose that P^* can be represented by a density function in E_n that is positive throughout some sphere centered at the point y^*. Then y^* is dominant and the relation R is transitive and completely orders all the points in E_n.*

The next theorem is a strong converse to Corollary 2 and establishes that whenever the relation R is transitive, then there must exist a dominant point.

THEOREM 5: *Suppose the relation R is transitive. Then there exists a dominant point $y^* \in E_n$.*

PROOF: In this proof it will be convenient to represent each point $y \in E_n$ in terms of its n coordinates $y = \{y_1, \ldots, y_n\}$. For $i = 1, \ldots, n$, we can consider the univariate distribution that is induced on the ith coordinate by the distribution P^* on E_n. Let m_i denote a median of this univariate distribution for $i = 1, \ldots, n$. We shall prove that the point $m = \{m_1, \ldots, m_n\}$ is dominant.

Now consider any vector $y \in E_n$ whose ith coordinate is m_i. Suppose that $z \in E_n$ is any other vector whose ith coordinate has some value other than m_i but for which all of the other $n - 1$ coordinates agree with the coordinates of y. Then it follows from Corollary 1 and the geometric interpretation of the relation R in E_n that yRz.

Therefore, if $y = \{y_1, \ldots, y_n\}$ is any point in E_n, the following relations must be satisfied:

$$\begin{aligned}
&\{m_1, m_2, \ldots, m_n\} R \{y_1, m_2, \ldots, m_n\}, \\
&\{y_1, m_2, \ldots, m_n\} R \{y_1, y_2, m_3, \ldots, m_n\}, \\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots \\
&\{y_1, y_2, \ldots, y_{n-1}, m_n\} R \{y_1, y_2, \ldots, y_{n-1}, y_n\}.
\end{aligned}$$

Since the relation R is transitive, we may conclude from this sequence of relations that mRy. Since y was an arbitrary point in E_n, it follows that m is a dominant point.

The preceding results establish the necessary and sufficient conditions for a transitive social preference ordering to be constructed from given individual orderings via the device of majority rule.

5. CONCLUDING COMMENTS

It should be obvious, of course, that this paper does not settle all of the issues raised by the appearance of Arrow's famous impossibility theorem. Even though the theorems that are stated and proved herein establish necessary and sufficient conditions for a transitive social preference ordering, the entire structure does rest upon the particular class of utility functions that were assumed. Although the authors find this class reasonable, and a suitable abstraction, the assumption does have somewhat less "generality" than is sometimes found in discussions of this kind.

If progress is to be made in our understanding of the problem of social choice, and if Arrow's impossibility theorem is not to be an impediment to further work on the topic, then we must continue to attempt to determine when such orderings can and cannot be constructed. We believe that the present work strongly suggests that considerations largely absent in the literature, such as the distribution of the preferences of the electorate, must be brought into the discussions if generalizations beyond the present one are to be established.

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Manuscript received October, 1969; revision received June, 1970.

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